Fundamental Math

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Contents

Chapter 1

Set Theory

1.1 Notation

1.1.1 Common Sets

Below is the notation that is typically used for several common sets:

1. $\mathbb{Z}$ – integers
2. $\mathbb{N}$ – natural numbers (non-negative integers)
3. $\mathbb{R}$ – real numbers
4. $\mathbb{R}_+$ – non-negative real numbers
5. $\mathbb{R}_{++}$ – strictly positive real numbers
6. $\mathbb{R}_-$ – non-positive real numbers
7. $\mathbb{R}_{--}$ – strictly negative real numbers
8. $\mathbb{C}$ – complex numbers
9. $\mathbb{C}_+$ – complex numbers with non-negative real part
10. $\mathbb{C}_{++}$ – complex numbers with strictly positive real part
11. $\mathbb{C}_-$ – complex numbers with non-positive real part
12. $\mathbb{C}_{--}$ – complex numbers with strictly negative real part
13. $\mathbb{F}$ – real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$
14. $\emptyset$ – null/empty set (set with no elements)
15. S – universal set (set of all possible elements in a given context)

1.1.2 Quantifiers

Below are quantifiers that are commonly used in set theory:

1. $\in$ – element of (a set)
2. $\notin$ – not an element of (a set)
3. $\subset$ – proper subset (of a set)
4. $\subseteq$ – subset of (a set)
5. $\not\subseteq$ – not a subset of (a set)
6. $\forall$ – for all
7. $\exists$ – there exists
8. $\exists!$ – there exists a unique
9. $\nexists$ – there does not exist
10. $\ni, |, :$ – such that (s.t.)
11. $\neg$ – not

1.2 Properties of Sets

1.2.1 Union and Intersection

The union of two sets A and B is denoted $A \cup B$ and is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The intersection of two sets A and B is denoted $A \cap B$ and is defined as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The complement of set A is denoted A^C and is defined as

$$A^C = \{x \in S : x \notin A\}.$$

De Morgan's law says that for two events A and B,

$$(A \cup B)^C = A^C \cap B^C.$$

1.2.2 Partition

A collection of sets $A_1, \ldots, A_n$ are **mutually exclusive/disjoint** if and only if

$$A_i \cap A_j = \emptyset \; \forall i \neq j.$$

A collection of sets $A_1, \ldots, A_n$ are **collectively exhaustive** if and only if

$$\bigcup_{i=1}^{n} A_i = S,$$

where S is the universal set. A **partition** is a collection of sets that are both mutually exclusive and collectively exhaustive.

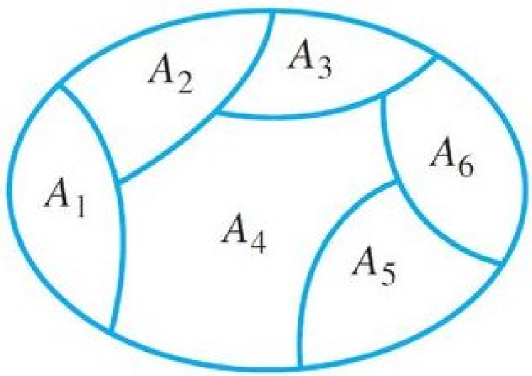

Figure 1.1: This is an example of a partition. The universal set, S, is the oval, which is composed of six disjoint and collectively exhaustive subsets, $A_1, \ldots, A_6$.

Chapter 2

Proofs

2.1 Implications

Below are implications that are commonly used in proofs:

$$A \Longrightarrow B \text{ – A implies B}$$
$$A \Longleftarrow B \text{ – B implies A}$$
$$A \Longleftrightarrow B \text{ – A if and only if B}$$

2.2 Types of Proofs

Below are a list of five common types of proofs:

1. **Direct Proof** – Logically derive the conclusion $A \Rightarrow B$ directly from relevant definitions, assumptions, lemmas, theorems, etc.

   Example: Prove that the sum of two odd numbers is even.

   Consider two integers $x, y \in \mathbb{Z}$, where $x = 2k_1 + 1$ for some $k_1 \in \mathbb{Z}$ and $y = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}$. We can show that $s := x + y$ is even, using the following definition: a number $n \in \mathbb{Z}$ is even if $n = 2k$ for some $k \in \mathbb{Z}$.

   $$s = (2k_1+1)+(2k_2+1) = 2k_1+2k_2+2 = 2(k_1+k_2+1) =: 2k_3, \text{where } k_3 \in \mathbb{Z}$$

   Therefore, s is even, proving that the sum of two odd numbers is even.

2. **Proof by Contrapositive** – Establish the conclusion $A \Rightarrow B$ by demonstrating that $\neg B \Rightarrow \neg A$. This is useful if it is easier to work with $\neg B$.

   Example: Show that if x^2 is even, then x is even.

   We can derive this conclusion by showing that if x is odd, then x^2 is odd. Suppose $x = 2k + 1$ for some $k_1 \in \mathbb{Z}$. Then,

   $$x^2 = (2k_1+1)^2 = 4k_1^2+4k_1+1 = 2(2k_1^2+2k_1))+1 =: 2k_2+1, \text{where } k_2 \in \mathbb{Z}$$

Therefore, x^2 is odd if x is odd. Because an integer must either be even or odd, this allows us to say that x is even if x^2 is even.

Example: Prove that not all odd numbers are prime.

3. **Proof by Contradiction/Negation** – Assume the given is true but the conclusion we want to show is false (i.e. A is true and B is false). Then show that the result contradicts this assumption. This then implies that the conclusion B must actually be true.

   Example: Prove that $\sqrt{2}$ is an irrational number.

   We can show this by finding a contradiction to the assumption that $\sqrt{2}$ is a rational number. We can use the following definition: a number q is rational if there exist $a, b \in \mathbb{Z}$ such that $q = a/b$, where $b \neq 0$. Assume that $\sqrt{2}$ is rational, meaning that $\exists\, a, b \in \mathbb{Z}$ such that $\sqrt{2} = a/b$. Without loss of generality, assume that a and b have no common factors.

   $$\sqrt{2} = a/b \implies a^2 = 2b^2$$

   Therefore, a^2 is even, which implies that a is even, so $a = 2k_1$ for $k_1 \in \mathbb{Z}$.

   $$(2k_1)^2 = 2b^2 \implies b^2 = 2k_1^2$$

   Therefore, b^2 is even, which implies that b is also even. Because a and b are both even, they have a common factor of 2. We have found a contradiction to our assumption that $\sqrt{2}$ is a rational number whose fraction in the simplest form is a/b. Therefore, $\sqrt{2}$ is irrational.

4. **Proof by Construction** – Construct an example that shows the fallacy or validity of a statement. This is usually useful for disproving an assertion such as "all X are Y" or for confirming a statement such as "there exists an X such that Y."

   Example: Prove that not all odd numbers are prime.

   Consider the number 9, which is odd because $9 = 2(4) + 1$. The number 9 has the factors 1, 3, and 9, so it is not prime. Therefore, not all odd numbers are prime.

5. **Proof by Induction** – First, prove the conclusion is true for an initial condition (e.g. $n = 1$). Then, show that if the conclusion is true for $n = k$, then it must also be true for $n = k + 1$.

   Example: Prove that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ for all $n \geq 1$.

   First consider the initial case, where n is one:

   $$\sum_{i=1}^{1} i = 1 = \frac{(1)(1+1)}{2} \checkmark$$

Now suppose that this equality holds for $n = k$:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

We can show that it also holds for $n = k + 1$:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

Therefore, by mathematical induction, this equality holds for all $n \geq 1$.